

Global aspects of p -branes

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Abstract

We generalize to dimension $p > 1$ the notion of string structure and discuss the related obstruction. We apply our results to a model of bosonic p -branes propagating on a principal G -bundle, coupled to a Yang–Mills field and an antisymmetric tensor field and in the presence of a Wess–Zumino term in the Lagrangian. We construct the quantization line bundle and discuss the action of background gauge transformations on wave functions.

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1. Introduction

The notion of string structure is an elegant way of formulating the absence of certain anomalies for superstrings coupled to gauge fields in the target space [1,2]. In this paper we will discuss the generalization of this notion to p -branes with arbitrary odd p .

Let us begin by choosing a p -dimensional manifold Σ which is compact, connected, oriented and without boundary, and a “target space” M , to be interpreted as spacetime. A p -brane is a map from Σ to M . In order to describe the coupling of the p -brane to a Yang–Mills field, we choose a principal bundle P with compact structure group G and base space M , with connection defined by some connection form α . We adopt the standard convention that G acts freely on P from the right. Let ΣM be a shorthand

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for $\text{Maps}(\Sigma, M)$, and similarly with P and G ('Maps' will always mean the space of smooth maps between the given domain and target). The configuration space of the " p -branes with internal symmetry", ΣP , is a principal bundle over ΣM with structure group ΣG . The right action of $g \in \Sigma G$ on $\varphi \in \Sigma P$ is given by the pointwise action of G on P :

$$(\varphi g)(\sigma) = \varphi(\sigma)g(\sigma) . \quad (1.1)$$

In the case of the string, the loop group S^1G has central extensions $\widehat{S^1G}$. A string structure is a prolongation of the principal S^1G bundle S^1P over S^1M to a principal $\widehat{S^1G}$ bundle $\widehat{S^1P}$. Note that $\widehat{S^1P}$ is also a circle bundle over S^1P .

In the case $p > 1$, p odd, the group ΣG also has extensions, but unlike the case of the string the interesting extensions are not simply central extensions. In the setting described above, the natural extensions are parametrized by connections in P and have as fiber the space of S^1 -valued functionals on ΣP . We will denote by $\widehat{\Sigma G}_\alpha$ the extension defined by the connection α . (These extensions will be described more precisely in Sections 2–4.) A priori there seem to be two natural ways of generalizing the notion of string structure to higher p -branes. One could define a " p -brane structure" as a prolongation of ΣP to a principal bundle $\widehat{\Sigma P}$ over ΣM with structure group $\widehat{\Sigma G}_\alpha$; or one may define it as a principal S^1 bundle $\widetilde{\Sigma P}$ over ΣP , together with a faithful right action of $\widehat{\Sigma G}_\alpha$ on $\widetilde{\Sigma P}$.

Both notions agree in the case $p = 1$. We shall follow here the second approach. Note that $\widetilde{\Sigma P}$, as a fiber bundle over ΣM , has a standard fiber isomorphic to a circle bundle over ΣG and is not a principal bundle.

The obstruction to our construction of a " p -brane structure" is given by a certain characteristic class of the bundle P , which depends on the class of the extension $\widehat{\Sigma G}_\alpha$. For the extensions that we shall consider, the obstruction is

$$c = k_p \text{tr } \phi^{(p+3)/2} , \quad (1.2)$$

where $\phi = d\alpha + \frac{1}{2}[\alpha, \alpha]$ is the curvature form of α , tr is the trace in a fixed finite dimensional representation of G and k_p is a normalization factor chosen in such a way that the integral of c over any $(p+3)$ -dimensional compact manifold without boundary is an integer. (If $G = SU(N)$, c is the Chern class $c_{(p+3)/2}$.) Other polynomial invariants can also be used, leading to different group extensions and prolongations.

In the case of the string, the existence of a string structure can be viewed as a condition for the existence of the Dirac–Ramond operator [1]. This operator is related to the supersymmetry charge. It is reasonable to expect that also in the case $p > 1$ the existence of a " p -brane structure" is a condition for the existence of an analogue of the Dirac–Ramond operator. This operator presumably arises in a supersymmetric theory of p -branes in which the degrees of freedom in the fibers of P are replaced by suitable chiral fermions. However, in the absence of a supersymmetric Lagrangian for p -branes coupled to Yang–Mills, we shall not discuss these aspects here. Instead, we shall consider a particular model dynamics for bosonic p -branes which was recently discussed [3]. At the local level it involves, in addition to the Yang–Mills field A , also a

$(p + 1)$ -form B . Invariance under target space gauge transformations (automorphisms of P) results in this model from a kind of Green–Schwarz anomaly cancellation mechanism in which the form B plays a crucial role. Unlike the case $p = 1$, this bosonic theory is not expected to be equivalent to its fermionic counterpart, but it reproduces faithfully features related to the cancellation of anomalies. We find that the construction of the invariant action requires that the characteristic class c vanishes. This is the manifestation at the Lagrangian level of the obstruction discussed above.

Starting from the invariant action, we construct the circle bundle $\widetilde{\Sigma P}_{(\alpha, H)}$ with right $\widehat{\Sigma G}_\alpha$ action (the construction depends on one additional datum, namely the $(p + 2)$ -form H given in (5.1) below). This bundle is the quantization bundle; the sections of the associated complex line bundle are the Schrödinger wave functions. We then show that the group of target space gauge transformations (automorphisms of P) can be realized on wave functions without extension. Our findings agree with the results of explicit canonical calculations in the case of a trivial bundle $P = M \times G$ [4,5].

2. Extensions of the Lie algebra $\Sigma\mathfrak{g}$

Let \mathfrak{g} be the Lie algebra of G and $\Sigma\mathfrak{g} = \text{Maps}(\Sigma, \mathfrak{g})$ the Lie algebra of ΣG , under pointwise commutators. Also, denote by \mathcal{A} the space of \mathfrak{g} -valued one-forms on Σ . There exist nontrivial extensions of $\Sigma\mathfrak{g}$ by the abelian ideal $\text{Maps}(\mathcal{A}, i\mathbb{R})$ [6,7]. These extensions can be described as follows. As a vector space, the extended Lie algebra is the direct sum $\Sigma\mathfrak{g} \oplus \text{Maps}(\mathcal{A}, i\mathbb{R})$. The Lie bracket is then given by

$$[(X, \gamma), (X', \gamma')] = ([X, X'], \delta_X \gamma' - \delta_{X'} \gamma + c_2(\cdot; X, X')) . \tag{2.1}$$

Here c_2 is a two-cocycle in $\Sigma\mathfrak{g}$ with values in $\text{Maps}(\mathcal{A}, i\mathbb{R})$, *i.e.* it satisfies:

$$\delta_X c_2(\cdot; Y, Z) - c_2(\cdot; [X, Y], Z) + \text{cyclic permutations} = 0 , \tag{2.2}$$

where δ_X denotes the infinitesimal gauge variation of a functional of A . The cocycle can be written as

$$c_2(A; X, Y) = \int_{\Sigma} \omega_p^2(A; X, Y) , \tag{2.3}$$

where ω_p^2 is a p -form on Σ depending polynomially on the vector potential $A \in \mathcal{A}$. The form ω_p^2 can be obtained by the dimensional descent procedure from an invariant polynomial in the curvature. Starting from the invariant polynomial given in (1.2), with ϕ replaced by $F = dA + \frac{1}{2}[A, A]$, one defines the Chern–Simons form $\omega_{p+2}^0(A)$ by

$$d\omega_{p+2}^0(A) = k_p \text{tr} F^{(p+3)/2} . \tag{2.4}$$

The superscript 0 refers to the degree of ω as a cochain on the Lie algebra $\Sigma\mathfrak{g}$, while the subscript $(p + 2)$ refers to its degree as a differential form. The coboundary of the Chern–Simons form defines (up to an exact form) the one-cochain ω_{p+1}^1 :

$$\delta_X \omega_{p+2}^0(A) = d\omega_{p+1}^1(A, X) . \quad (2.5)$$

It can be written as

$$\omega_{p+1}^1(A, X) = \text{tr } dX \Phi_p(A) , \quad (2.6)$$

where the p -form $\Phi_p = \Phi_p^a T_a$ is a polynomial in A and F . For $p = 1, 3, 5$ this polynomial is given by

$$\Phi_1 = -k_1 A , \quad (2.7a)$$

$$\Phi_3 = -\frac{1}{2} k_3 (FA + AF - A^3) , \quad (2.7b)$$

$$\begin{aligned} \Phi_5 = & -\frac{1}{3} k_5 [(F^2 A + FAF + AF^2) - \frac{4}{5} (A^3 F + FA^3) \\ & - \frac{2}{5} (A^2 FA + AFA^2) + \frac{3}{5} A^5] . \end{aligned} \quad (2.7c)$$

The coboundary of ω_{p+1}^1 defines ω_p^2 :

$$\delta_X \omega_{p+1}^1(A, Y) - \delta_Y \omega_{p+1}^1(A, X) - \omega_{p+1}^1(A, [X, Y]) = d\omega_p^2(A, X, Y) . \quad (2.8)$$

In the cases $p = 1, 3, 5$ we have

$$\omega_1^2(A, X, Y) = -2k_1 \text{tr } XdY , \quad (2.9a)$$

$$\omega_3^2(A, X, Y) = -k_3 \text{tr} [dX, dY] A , \quad (2.9b)$$

$$\omega_5^2(A, X, Y) = \frac{1}{15} k_5 \text{tr} (5F - 3A^2) (2A[dX, dY] - dXAdY + dYAdX) . \quad (2.9c)$$

These are the forms that we shall use in the definition of the cocycles (2.3). In particular for $p = 1$ this gives the familiar central term of a Kac–Moody algebra; since it is independent of A in this special case the fiber can be restricted to the constant functions.

It will be useful to consider also the cocycles \hat{c}_2 differing from c_2 by the coboundary of the one-cochain $\Phi(A, X) = \int_{\Sigma} \text{tr } X\Phi_p(A)$:

$$\hat{c}_2(A, X, Y) = c_2(A, X, Y) - (\delta_X \Phi(A, Y) - \delta_Y \Phi(A, X) - \Phi(A, [X, Y])) . \quad (2.10)$$

These cocycles can also be written in the form (2.3), with

$$\hat{\omega}_1^2(A, X, Y) = k_1 \text{tr} [X, Y] A , \quad (2.11a)$$

$$\hat{\omega}_3^2(A, X, Y) = \frac{1}{2} k_3 \text{tr} ([X, Y] (FA + AF - A^3) + XdAYA - XAYdA) , \quad (2.11b)$$

$$\begin{aligned} \hat{\omega}_5^2(A, X, Y) = & \frac{1}{5} k_5 \text{tr} \left\{ [X, Y] [2(F^2 A + AF^2) + FAF - A^3 F - FA^3 \right. \\ & - A^2 FA - AFA^2] + 2(XdAY - YdAX) (FA + AF - A^3) \\ & \left. - 2(XAY - YAX)d(FA + AF - A^3) \right\} . \end{aligned} \quad (2.11c)$$

Let us now fix a connection in P with connection form α . The pullback of α by means of the map φ is an element of \mathcal{A} and the right action (1.1) of ΣG on ΣP induces a gauge transformation on the potential $\varphi^* \alpha$:

$$(\varphi g)^* \alpha = g^{-1} (\varphi^* \alpha) g + g^{-1} dg . \quad (2.12)$$

Consider the two-cocycle c'_2 with values in $\text{Maps}(\Sigma P, i\mathbb{R})$ defined by

$$c'_2(\varphi; X, Y) = c_2(\varphi^* \alpha; X, Y) . \quad (2.13)$$

(In the following we will drop the primes for notational simplicity.) It gives rise to an extension of $\Sigma\mathfrak{g}$ by $\text{Maps}(\Sigma P, i\mathbb{R})$, denoted by $\widehat{\Sigma\mathfrak{g}}_\alpha$. These are the extensions arising in p -brane theory.

3. Cohomology of ΣP

It will be useful to view the cocycles (2.13) from a different point of view. We recall that given an element X of the Lie algebra \mathfrak{g} of G one can construct a vector field \tilde{X} on P , called a fundamental vector field, generating the right action of G . This map is an isomorphism from \mathfrak{g} to the vertical subspace at each point in P . Given a k -form β_k on ΣP , one can construct a k -cochain c_k on $\Sigma\mathfrak{g}$ with values in $\text{Maps}(\Sigma P, i\mathbb{R})$ by

$$c_k(\varphi; X_1, \dots, X_k) = (\beta_k(\varphi))(\tilde{X}_1, \dots, \tilde{X}_k) . \tag{3.1}$$

Conversely, given a k -cochain c_k one can always find a k -form β_k on ΣP satisfying (3.1). Clearly this form is not uniquely defined: only the restriction of β_k to the fibers of ΣP is determined by this condition. We thus have a surjective map from k -forms on ΣP to k -cochains on $\Sigma\mathfrak{g}$ with values in $\text{Maps}(\Sigma P, i\mathbb{R})$. The kernel of this map consists of the forms whose contraction with a vertical vector vanishes. Under this map, the exterior differential of β_k is mapped to the coboundary of c_k . Closed (exact) forms are mapped to cocycles (coboundaries). However, note that, if c_k is a cocycle, there are forms β_k satisfying (3.1) which are not closed. All that is required of $d\beta_k$ is that its restriction to the fibres vanishes.

We would like now to find a two-form on ΣP which is related to the the two-cocycle (2.13) by the map defined above. This will require some other preliminaries. We recall first that there is a map E from $(k + p)$ -forms on M to k -forms on ΣM given by

$$(E\gamma)_\varphi(v_1, \dots, v_k) = \int_\Sigma \varphi^* \gamma(v_1, \dots, v_k, \cdot, \dots, \cdot) . \tag{3.2}$$

The same map can be defined with M replaced by P or G . This map commutes with the exterior differential and hence defines a map of cohomology classes. It is also dual to the evaluation map in the sense that $\int_{N_k} E\gamma = \int_{ev(N_k)} \gamma$, where N_k is a k -dimensional submanifold of ΣM and its evaluation is a $(k + p)$ -dimensional submanifold of M . Due to this property, E maps integral cocycles to integral cocycles.

Let us recall also the definition of the transgression in the bundle P . Consider a closed k -form β_G in the fiber G which is the restriction of a k -form β in P such that $d\beta = \pi^*\gamma$ for some $(k + 1)$ form γ in M . Then one says that $[\gamma] \in H^{k+1}(M)$ is the transgression of $[\beta_G] \in H^k(G)$. The classic example of transgression is provided by the Chern–Simons form $\omega_{p+2}^0(\alpha)$. Since the restriction of a connection form α to a fiber coincides with the left-invariant Maurer–Cartan form $g^{-1}dg$, the restriction of $\omega_{p+2}^0(\alpha)$ to a fiber is

$$\sigma = \omega_{p+2}^0(g^{-1}dg) = k_p a_p \text{tr}(g^{-1}dg)^{p+2} , \tag{3.3}$$

where

$$a_p = (-1)^{(p+1)/2} \left(\frac{p+3}{2} \right) \Gamma \left(\frac{p+3}{2} \right)^2 / \Gamma(p+3).$$

From (2.4) (with α and ϕ replacing A and F) there follows that $[\sigma] \in H^{p+2}(G)$ transgresses to $[c] \in H^{p+3}(M)$.

Our construction will be based on the image of this argument under the map E . The only complication is that the restriction of $E\omega_{p+2}^0(\alpha)$ to a fiber of ΣP is in general not equal to $E\sigma$, since the differentials of a general map φ are not vertical. For this reason, it will be convenient to choose a basepoint φ_0 in ΣP which maps Σ to the fiber of P over the basepoint x_0 of M . Thus the composition of φ_0 with the projection $\pi : P \rightarrow M$ is the constant map x_0 . We will call the fiber through φ_0 the “typical fiber”, and identify it with the group ΣG . The restriction of $E\omega_{p+2}^0(\alpha)$ to the typical fiber is equal to $E\sigma$, because the differentials of maps in the typical fiber are vertical in P .

A direct calculation shows that under the map defined by (3.1) the two-form $E\omega_{p+2}^0$ is mapped to the cocycle $-\hat{c}_2$. In fact, using that on a fundamental vectorfield $\alpha(\tilde{X}) = X$ and $\phi(\tilde{X}, \cdot) = 0$, one finds

$$\begin{aligned} & (E\omega_{p+2}^0(\alpha))_{\varphi}(\tilde{X}, \tilde{Y}) \\ &= -(p+2)k_p a_p \int_{\Sigma} \varphi^* \text{tr} (XY\alpha^p - X\alpha Y\alpha^{p-1} + \dots - X\alpha^p Y) \\ &= -\hat{c}_2(\varphi^* \alpha; X, Y). \end{aligned} \tag{3.4}$$

It was shown in (2.10) that \hat{c}_2 is cohomologous to c_2 . We can now find a two-form ψ which is related to the cocycle c_2 . Consider the one-form \mathcal{T} on ΣP defined by $\mathcal{T}(v) = \int_{\Sigma} \text{tr} \alpha(v) \Phi_p(\alpha)$. When evaluated on a fundamental vectorfield, this form is seen to correspond to the cochain Φ . Therefore, the form

$$\psi = -(E\omega_{p+2}^0(\alpha) + d\mathcal{T}) \tag{3.5}$$

is related to the cocycle c_2 as in (3.1). We emphasize at this point that the form ψ is neither closed nor integral. However, its restriction to the fibers is closed and integral. It is determined by c_2 only up to a form whose contraction with a vertical vector field is zero. In particular, it is defined up to a basic form.

In all of the above the connection α in P was kept fixed. We can now prove that the cohomology class of the extension does not depend upon the choice of connection. Let ψ and ψ' be constructed as above starting from two different connections α and α' . The difference $\psi' - \psi$ is equal to $E(\omega_{p+2}^0(\alpha') - \omega_{p+2}^0(\alpha))$ plus an exact form. From Eq. (2.4) and the definition of the Weil homomorphism it follows that $\omega_{p+2}^0(\alpha') - \omega_{p+2}^0(\alpha)$ is the sum of a basic and a closed form. Using the properties of E , also $E(\omega_{p+2}^0(\alpha') - \omega_{p+2}^0(\alpha))$ is the sum of a basic and a closed form. Therefore the restriction of $\psi' - \psi$ to a fiber of ΣP is closed. In fact, it has to be exact. This is because the restriction of $\omega_{p+2}^0(\alpha') - \omega_{p+2}^0(\alpha)$ to any fiber of P is zero, and therefore the restriction of $E(\omega_{p+2}^0(\alpha') - \omega_{p+2}^0(\alpha))$ to the typical fiber defined above is also zero. Forms in ΣG

obtained by restriction of a form on ΣP to different fibers have to be cohomologous. Therefore the restriction of $\psi' - \psi$ to any fiber of ΣP has to be exact. This proves that the cocycles $c_2(\varphi^* \alpha)$ and $c_2(\varphi^* \alpha')$ are cohomologous.

4. Extensions of ΣG

We are now going to assume that ΣG is simply connected. This is the case if Σ is the unit sphere S^p and G is connected, simply connected and has $\pi_{p+1}(G) = 0$ (the most important example being $G = SU(N)$ with $N \geq (p + 3)/2$). Then, there is a one-to-one correspondence between extensions of the Lie algebra $\Sigma \mathfrak{g}$ and extensions of the Lie group ΣG . The extensions of $\Sigma \mathfrak{g}$ by $Maps(\mathcal{A}, i\mathbb{R})$ give rise to topologically nontrivial extensions of ΣG by the abelian group $Maps(\mathcal{A}, S^1)$, which have been described in Ref. [7]. We are now going to describe the extension $\widehat{\Sigma G}_\alpha$ corresponding to the Lie algebra $\widehat{\Sigma \mathfrak{g}}_\alpha$.

We begin by defining $N = \Sigma \times [0, 1]$ and fix a basepoint in each connected component of ΣG . We assume that the basepoints have been chosen in such a way that the product of two basepoints is another basepoint. The basepoint in the connected component of the identity has to be the identity map. One can extend every map $g : \Sigma \rightarrow G$ to a map $\hat{g} : N \rightarrow G$, such that $g(\sigma, 1) = g(\sigma)$ and $g(\sigma, 0)$ is the basepoint in the connected component containing g . One can also think of \hat{g} as a path in ΣG beginning at the basepoint and ending at g . Consider pairs (\hat{g}, λ) , where $\hat{g} \in NG \equiv Maps(N, G)$ and $\lambda \in Maps(\Sigma P, S^1)$. We define an equivalence relation on these pairs: $(\hat{g}, \lambda) \sim (\hat{g}', \lambda')$ if the two paths have the same endpoints $\hat{g}(1) = \hat{g}'(1) = g$ and $\lambda'(\varphi) = \lambda(\varphi) \exp(2\pi i \int_S \psi)$, where $S = S[\varphi, \hat{g}, \hat{g}']$ is a two-dimensional surface in ΣP bounded by the paths $\varphi \hat{g}$ and $\varphi \hat{g}'$, both originating at φ and ending at φg . Note that one can choose S to lie entirely in the fiber through φ and therefore the integral is not affected by the arbitrariness in ψ . The group $\widehat{\Sigma G}_\alpha$ consists of these equivalence classes of pairs, with the multiplication:

$$[(\hat{g}_1, \lambda_1)][(\hat{g}_2, \lambda_2)] = [(\hat{g}_1 \hat{g}_2, \lambda_1(\hat{g}_1 \cdot \lambda_2) e^{2\pi i \theta_2(\cdot, \hat{g}_1, \hat{g}_2)})] , \tag{4.1}$$

where θ_2 is a suitable functional of φ , \hat{g}_1 and \hat{g}_2 and the left action of NG on $Maps(\Sigma P, S^1)$ is defined by $(\hat{g} \cdot \lambda)(\varphi) = \lambda(\varphi g)$. For associativity, θ_2 has to be a two-cocycle:

$$\theta_2(\varphi \hat{g}_1, \hat{g}_2, \hat{g}_3) - \theta_2(\varphi, \hat{g}_1 \hat{g}_2, \hat{g}_3) + \theta_2(\varphi, \hat{g}_1, \hat{g}_2 \hat{g}_3) - \theta_2(\varphi, \hat{g}_1, \hat{g}_2) \equiv 0 \pmod{\mathbb{Z}} . \tag{4.2}$$

In order to reproduce the infinitesimal cocycles (2.4), the phase has to be chosen as

$$\theta_2(\varphi, \hat{g}_1, \hat{g}_2) = \int_K \psi , \tag{4.3}$$

where K is the two-dimensional simplex in ΣP with vertices in φ , φg_1 , $\varphi g_1 g_2$ and bounded by the curves $\varphi \hat{g}_1$, $\varphi \hat{g}_1 \hat{g}_2$ and $\varphi g_1 \hat{g}_2$ (with the obvious notation $(\hat{g}_1 \hat{g}_2)(t) =$

$\hat{g}_1(t)\hat{g}_2(t)$). The associativity is then automatically satisfied. In fact if we call τ a locally defined one-form in the fiber through φ such that $d\tau$ coincides with the restriction of ψ and define $\theta(\varphi, \hat{g})$ to be the line integral $\int_{\varphi\hat{g}} \tau$ along the path $\varphi\hat{g}$, then

$$\theta_2(\varphi, \hat{g}_1, \hat{g}_2) = \theta(\varphi\hat{g}_1, \hat{g}_2) - \theta(\varphi, \hat{g}_1\hat{g}_2) + \theta(\varphi, \hat{g}_1) . \tag{4.4}$$

The proof that the multiplication (4.1) is independent of the representatives on the l.h.s. boils down to showing that

$$\int_{S[\varphi, \hat{g}_1\hat{g}_2, \hat{g}'_1\hat{g}_2]} \psi - \int_{S[\varphi, \hat{g}_1, \hat{g}'_1]} \psi - \theta_2(\varphi, \hat{g}'_1, \hat{g}_2) + \theta_2(\varphi, \hat{g}_1, \hat{g}_2) = 0 \pmod{\mathbb{Z}} , \tag{4.5}$$

$$\int_{S[\varphi, \hat{g}_1, \hat{g}_2, \hat{g}'_2]} \psi - \int_{S[\varphi, \hat{g}_1\hat{g}_2, \hat{g}_1\hat{g}'_2]} \psi - \theta_2(\varphi, \hat{g}_1, \hat{g}_2) + \theta_2(\varphi, \hat{g}_1, \hat{g}'_2) = 0 \pmod{\mathbb{Z}} . \tag{4.6}$$

These relations are indeed true for θ_2 defined as in (4.3), since their l.h.s. are the integrals of ψ on closed two-dimensional submanifolds in a fiber of ΣP , and the restriction of ψ to the fiber is closed and integral. Note also that θ_2 is not affected by the arbitrariness in ψ which was discussed in the previous section.

The fiber bundles $\widehat{\Sigma G}_\alpha \rightarrow \Sigma G$ may or may not be trivial; this depends on the choice of G and Σ . For example, when $G = SO(N)$ or any closed subgroup of $SO(N)$ and $p = 4n - 1$ the bundles become trivial. The reason is simply that the characteristic classes $\text{tr} \phi^{2n+1}$ vanish identically when the curvature ϕ takes values in the Lie algebra of antisymmetric real matrices. On the other hand, when $G = SU(N)$ and $p \leq 2N - 3$ then the characteristic classes $\text{tr} \phi^{(p+3)/2}$ are non trivial. In these cases the Chern–Simons form $\omega_{p+2}^0(\alpha)$, when restricted to a fiber of P , coincides with the form σ defined in (3.3) (the WZWN anomaly); this in turn implies that the form ψ used in the definition of the extension is topologically nontrivial.

5. The circle bundle $\widehat{\Sigma P}_{(\alpha, H)}$

In the introduction we have defined a “ p -brane structure” in P to be a circle bundle over ΣP together with a faithful right action of $\widehat{\Sigma G}_\alpha$ covering the action of ΣG on ΣP . We will now give a construction of a “ p -brane structure” based on the assumption that the characteristic class (1.2) vanishes. It seems plausible that $[c] = 0$ is also a necessary condition for the existence of such a structure, but we shall not investigate this question here.

Assume there exists a basic $(p + 2)$ -form H on P such that $c = dH$. Then

$$\Omega = \omega_{p+2}^0(\alpha) - H \tag{5.1}$$

is a closed $(p + 2)$ -form on P . One can choose H so that Ω is also integral [8]. From (3.5) and the properties of E we find that the two-form Θ in ΣP defined by

$$\Theta = -(E\Omega + dT) = \psi + EH \tag{5.2}$$

is closed, integral and is related to c_2 by the map (3.1). If ΣP is simply connected, Θ can be used to construct a circle bundle with connection $\widetilde{\Sigma P}_{(\alpha,H)}$ over ΣP .

Again we denote $N = \Sigma \times [0, 1]$ and choose a basepoint in each connected component of ΣP . We assume that each connected component of ΣP is simply connected. We extend the maps $\varphi : \Sigma \rightarrow P$ to maps $\hat{\varphi} : N \rightarrow P$, which can be regarded as paths in ΣP from the basepoint φ_0 to φ . The total space of the bundle $\widetilde{\Sigma P}_{(\alpha,H)}$ is $(Maps(N, P) \times S^1) / \sim$, where \sim is the equivalence relation given by

$$(\hat{\varphi}, \lambda) \sim (\hat{\varphi}', \lambda e^{2\pi i \int_S \Omega}), \tag{5.3}$$

where $\hat{\varphi}(1) = \hat{\varphi}'(1) = \varphi$ and $S = S[\hat{\varphi}, \hat{\varphi}']$ is a two-dimensional submanifold of ΣP bounded by the paths $\hat{\varphi}$ and $\hat{\varphi}'$. Such a manifold always exists since we have assumed that ΣP is simply connected and therefore any two homotopies with the same endpoints can be deformed into each other. We denote points of $\widetilde{\Sigma P}_{(\alpha,H)}$ by equivalence classes of pairs $[(\hat{\varphi}, \lambda)]$.

We define a right action of $\widehat{\Sigma G}_\alpha$ on $\widetilde{\Sigma P}_{(\alpha,H)}$ by

$$[(\hat{\varphi}, \lambda)] \cdot [(\hat{g}, \mu)] = [(\hat{\varphi} \cdot \hat{g}, (\lambda \cdot \hat{g}) \mu e^{2\pi i \chi(\hat{\varphi}, \hat{g})})] . \tag{5.4}$$

The phase factor is defined by

$$\chi(\hat{\varphi}, \hat{g}) = \int_K \Theta , \tag{5.5}$$

where $K = K[\hat{\varphi}, \hat{g}]$ is the two-simplex in ΣP with vertices in $\varphi_0, \varphi, \varphi g$ and bounded by the paths $\hat{\varphi}, \varphi \hat{g}$ and $\hat{\varphi} \hat{g}$. Note that, unlike in Section 4, the phase is now defined as an integral of Θ rather than ψ . This is because the surface K does not lie in a fiber of ΣP and ψ was closed and integral only when restricted to a fiber.

To prove that this action is independent of the representatives on the l.h.s. one has to show that

$$\chi(\hat{\varphi}', \hat{g}) - \chi(\hat{\varphi}, \hat{g}) - \int_{S[\hat{\varphi}\hat{g}, \hat{\varphi}'\hat{g}]} \Theta + \int_{S[\hat{\varphi}, \hat{\varphi}']} \Theta = 0 \pmod{\mathbb{Z}} , \tag{5.6}$$

$$\chi(\hat{\varphi}, \hat{g}') - \chi(\hat{\varphi}, \hat{g}) - \int_{S[\hat{\varphi}\hat{g}, \hat{\varphi}\hat{g}']} \Theta + \int_{S[\varphi\hat{g}, \varphi\hat{g}']} \Theta = 0 \pmod{\mathbb{Z}} . \tag{5.7}$$

These relations are indeed true, since their left hand sides are the integrals of Θ on closed two-dimensional submanifolds of ΣP .

Finally we have to show that this action is compatible with the group multiplication given in (4.1). Acting on $[(\hat{\varphi}, \mu)]$ with $[(\hat{g}_1, \lambda_1)]$ and then with $[(\hat{g}_2, \lambda_2)]$, and comparing with the action of the product of these group elements, we are led to the condition that the coboundary of χ has to be θ_2 :

$$\chi(\hat{\varphi}\hat{g}_1, \hat{g}_2) - \chi(\hat{\varphi}, \hat{g}_1\hat{g}_2) - \chi(\hat{\varphi}, \hat{g}_1) = \theta_2(\varphi, \hat{g}_1, \hat{g}_2) \pmod{\mathbb{Z}} . \tag{5.8}$$

This condition is again true, as one can easily verify by considering the tetrahedron in ΣP with vertices in $\varphi_0, \varphi, \varphi g_1, \varphi g_1 g_2$ and with edges $\hat{\varphi}, \hat{\varphi}\hat{g}_1, \hat{\varphi}\hat{g}_1\hat{g}_2, \varphi\hat{g}_1, \varphi\hat{g}_1\hat{g}_2, \varphi g_1\hat{g}_2$.

The three faces touching φ_0 give the three terms on the l.h.s. The remaining face lies entirely in the fiber through φ and therefore the integral of Θ on that face is equal to the integral of ψ , thus reproducing (4.3).

6. A model of bosonic p -branes

Fix a riemannian metric g on M and an invariant positive definite bilinear form $\langle \cdot, \cdot \rangle$ in the Lie algebra \mathfrak{g} . For any principal connection in P one can construct a unique riemannian metric on P such that the following holds: i) the horizontal and vertical spaces are orthogonal, ii) the inner product of horizontal vectors is equal to the inner product of their projections to M and iii) the inner product of vertical vectors is equal to the inner product of the corresponding Lie algebra elements.

The manifold Σ is endowed with a riemannian metric γ . The time evolution of the p -brane is given by a map, still denoted φ , from $\Sigma \times I$ to P , where I is a time interval. The action is the sum of a kinetic and a topological part: $S = S_{\text{kin}} + S_{\text{top}}$. The kinetic term is

$$S_{\text{kin}} = \frac{1}{2} \int_{\Sigma \times I} d^p \sigma dt \sqrt{-\det \gamma} \gamma^{ij} (\partial_i \varphi, \partial_j \varphi)_P, \tag{6.1}$$

where $(\cdot, \cdot)_P$ is the riemannian inner product defined above and the riemannian metric γ of Σ has been extended to a product Lorentzian metric on $\Sigma \times I$. In order to define the topological term we assume that space-time is compact and without boundary. This can be done by compactifying I to S^1 . Then

$$S_{\text{top}} = \int_{N \times S^1} \hat{\varphi}^* \Omega, \tag{6.2}$$

where Ω has to be a closed integral $(p + 2)$ -form on P . At each time the field φ is extended to a map $N \rightarrow P$ as before. We choose Ω to be as in (5.1).

To make contact with earlier work, we compute the form of the action with respect to a local trivialization of P . If (x, h) are local coordinates on P , the connection form is given by

$$\alpha(x, h) = h^{-1} dh + h^{-1} A(x) h, \tag{6.3}$$

where A , a locally defined form on M , is the Yang–Mills potential. Locally, the map φ can be represented by a pair of maps $x : \Sigma \rightarrow M$ and $h : \Sigma \rightarrow G$. The differential of φ can be decomposed into vertical and horizontal parts: $d\varphi = (dx, dh) = (0, dh + h \cdot x^* A) + (dx, -h \cdot x^* A)$. Inserting into (6.1),

$$S_{\text{kin}} = \frac{1}{2} \int_{\Sigma \times I} d^p \sigma dt \sqrt{-\det \gamma} \gamma^{ij} [g(\partial_i x, \partial_j x) + \langle h^{-1} \partial_i h + \partial_i x^\mu A_\mu, h^{-1} \partial_j h + \partial_j x^\mu A_\mu \rangle]. \tag{6.4}$$

For the topological term we observe that, using (6.3),

$$\omega_{p+2}^0(\alpha) = \omega_{p+2}^0(A) + dC(A, dhh^{-1}) + \omega_{p+2}^0(dhh^{-1}) , \tag{6.5}$$

where the $p + 1$ form C is a differential polynomial in the indicated arguments. Furthermore, we can define locally a $(p + 1)$ -form B on M such that

$$H = \omega_{p+2}^0(A) + dB . \tag{6.6}$$

Then

$$\Omega = \omega_{p+2}^0(dhh^{-1}) + dC - dB . \tag{6.7}$$

This is the topological term given in Ref. [3] (in comparing with Ref. [3] one has to take into account that here we are gauging the left action of G on itself whereas in Ref. [3] the right action was gauged, contrary to what is stated there).

Strictly speaking, in the case $p > 1$ the only invariance of the action is the finite dimensional group G acting on φ by right multiplication. This is because the connection α is not a dynamical variable and therefore should be treated as a fixed background. However, if one transforms also α , the action is also invariant under the group $\mathcal{G} = \text{Aut } P$ acting on φ by composition and on α and H by pullback (in particular, since H is basic, it is invariant).

In the Schrödinger picture the wave functions are sections of a complex line bundle \mathcal{L} over ΣP . The choice of the (equivalence class of the) line bundle is dictated by the action principle. In the present case the line bundle is obtained as the associated bundle to the circle bundle $\widetilde{\Sigma P}_{(\alpha,H)}$ through the natural representation of S^1 in \mathbb{C} . This can be checked by comparing the anomalies of the Poisson brackets of Noether charges associated to the symmetry group of right G multiplications on P and the commutator anomalies (2.11). It was found in Ref. [5] that in the case of a trivial bundle $P = M \times G$, the Poisson bracket algebra of the Noether charges associated to the right action of G on P has an extension given by the cocycle $c_2(h^{-1}dh)$. On the other hand, we recall from Section 3 that the cohomology class of the extension $\widehat{\Sigma g}_\alpha$ is independent of the connection α . Thus if P is a trivial bundle one can choose without loss of generality the flat connection for which α is equal to the left-invariant Maurer–Cartan form. This means that if one wants to lift the action of ΣG on ΣP to the quantization bundle, the group has to be extended to $\widehat{\Sigma G}_{h^{-1}dh}$. Thus the quantization bundle is $\widetilde{\Sigma P}_{(h^{-1}dh,H)}$. Essentially the same local computations apply, using local trivializations, in the general case.

Up to this point we have considered the construction of the circle bundle $\widetilde{\Sigma P}_{(\alpha,H)}$ for fixed α and H . Let \mathcal{M} be the set of all pairs (α, H) such that $\omega_{p+2}^0(\alpha) - H$ is closed and integral. Now consider the union of all bundles $\widetilde{\Sigma P}_{(\alpha,H)}$; since the construction of these bundles depended smoothly on the data (α, H) , we obtain a bundle $\widetilde{\Sigma P}$ over \mathcal{M} with fibers $\widetilde{\Sigma P}_{(\alpha,H)}$. We can represent points of $\widetilde{\Sigma P}$ as quadruples $(\hat{\varphi}, \lambda, \alpha, H)$, with the equivalence relation $(\hat{\varphi}, \lambda, \alpha, H) \sim (\hat{\varphi}', \lambda \exp(2\pi i \int_S \Omega), \alpha, H)$, where S is as in (5.3). Note that $\widetilde{\Sigma P}$ is also a circle bundle over $\Sigma P \times \mathcal{M}$.

The gauge group $\mathcal{G} = \text{Aut } P$ acts on ΣP by composition: for $u \in \mathcal{G}$, $(u \cdot \varphi)(\sigma) = u(\varphi(\sigma))$. This definition can be applied also to maps from N to P . The automorphisms act also on forms on P by pullback. In particular, this gives the usual action on connections, and a trivial action on basic forms such as H .

We would like to lift this action of \mathcal{G} on $\Sigma P \times \mathcal{M}$ to an action on $\widetilde{\Sigma P}$. The obvious definition of the action of \mathcal{G} is

$$u[(\hat{\varphi}, \lambda, \alpha, H)] = [(u \cdot \hat{\varphi}, \lambda, u^{-1*} \alpha, H)]. \quad (6.8)$$

If we apply this to equivalent quadruples we find that since $\omega_{p+2}^0(u^{-1*} \alpha) = u^{-1*} \times \omega_{p+2}^0(\alpha)$, also the transformed quadruples are equivalent. Therefore the definition given above goes to the quotient and defines an action of \mathcal{G} on $\widetilde{\Sigma P}$. In this case there is no extension, and at the infinitesimal level no commutator anomaly. Again this agrees with the result of Refs. [4,5] for a trivial bundle. Note that if one writes H locally in the form (6.6), the gauge variations of the two terms on the r.h.s. have to cancel. Using (2.5), the gauge variation of B has to be $\delta_X B = -\omega_{p+1}^1(A, X)$, up to an exact form.

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